

Fractional quantum Hall states as an Abelian group

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Abstract

We show that the set of double-layer Fractional Quantum Hall (FQH) states with a given topological order form a finite Abelian group under a new product. This group structure makes it possible to construct new FQH states from known ones. We also introduce a new index which can be used to characterize the topological order of FQH states.

Fractional quantum Hall (FQH) states [6, 5] (see [12] for an excellent introduction) represent novel phases of matter which arise from the interactions of electrons in 2D layers under the influence of strong magnetic fields (~ 30 Tesla) and at very low temperatures ($\sim 1K^\circ$). Since at very low temperature the interactions of electrons are strong, the FQH state is an example of a strongly correlated system. The new phases are not characterized by any continuous symmetry or a local order parameter as in the the usual Landau theory of symmetry breaking [8, 9, 10]. In [8, 9, 10], it was proposed that FQH states are characterized by a new kind of order, *topological order*. This order is not the result of local long range interactions between the elementary excitations but arise from the global dancing pattern of the electrons [7]. One measure of topological order is the number of ground states of the Hamiltonian [1]. This ground state degeneracy appears when one studies the FQH state on a genus- g Riemann surface, e.g., on a two torus which is equivalent to a layer with periodic boundary conditions in the two directions. The ground state degeneracy is a universal property and is robust against local perturbations [13].

The kind of order one gets in FQH states is different from crystal order. Crystal order is static and arises from the spatial configuration and location of the

atoms inside the crystal. However, the order in topological phases is dynamical and arises from the dancing patterns of the electrons in FQH states. This dancing pattern is specified by an integer symmetric matrix K [14]. In this note, we define an Abelian group structure on the set of $SL(2, \mathbb{Z})$ equivalence classes of K with a given determinant.

We will be interested in Abelian FQH states, where Abelian here means that the quasi-particle excitations follow Abelian statistics. The effective theory of such states is a $U(1)$ Chern-Simons gauge theory in $2 + 1$ dimensions described by the action [14]

$$S = -\frac{k}{4\pi} \int_M A \wedge dA + \frac{q}{2\pi} a \wedge dA, \quad (1)$$

where $k \in \mathbb{Z}$ is the level of the Chern-Simons term and A_μ is the gauge field which is dual to the quasiparticle conserved current, $J_\mu = \epsilon_{\mu\nu\alpha} \partial_\nu A_\alpha$. Here a_μ is an external $U(1)$ gauge field which couples to a quasiparticle excitations with charge q . One can also add a Yang-Mills term to the above action but the Yang-Mills term is an irrelevant operator in 3D and at low energies the Yang-Mills term is dominated by the Chern-Simons term.

The generalization of (1) to the multi-layer FQH states is described by a $U(1)^N$ Chern-Simons theory with an action

$$S = -\frac{1}{4\pi} K_{IJ} \int_M A^I \wedge dA^J + \frac{1}{2\pi} q_I a \wedge dA_I, \quad I = 1, \dots, N, \quad (2)$$

where K_{IJ} is a symmetric integer matrix and the index I characterizes the different condensates of the FQH state [12]. In the canonical quantization of the above action one first divide the three manifold M as $M = \mathbb{R} \times \Sigma_g$, where \mathbb{R} represents the time direction and Σ_g is the genus- g Riemann surface which represents the two-dimensional layer over which the FQH state lives. The quantum mechanical degrees of freedom are given by the harmonic parts of A_I on Σ_g and correspond to non-trivial global configurations of A_I . After quantization one gets a degenerate ground state with a degeneracy D given by

$$D = \det(K)^g. \quad (3)$$

The action (2) is supplemented with a quantization condition on the set of allowed charges of the quasiparticles. This quantization condition implies that the set of allowed charges live in an integer lattice and the gauge group is the compact $U(1)^N$. Due to this quantization condition, the allowed field redefinitions on the set of A^I are those transformations which preserve the integrality of the charges, i.e, $SL(N, \mathbb{Z})$ transformation

$$A_I \longrightarrow S_{IJ} A_J, \quad S \in SL(N, \mathbb{Z}). \quad (4)$$

In [2], the multi-layer FQH stat which is described by (2) was proposed

$$\Psi_K(z) = \prod_{a < b; I, J} (z_{aI} - z_{bJ})^{K_{ab}} \exp \left(- \sum_{a, I} |z_{aI}|^2 \right). \quad (5)$$

This state is labeled by the matrix K and a charge vector \mathbf{t} and describes a FQH state with a filling fraction¹

$$\nu = \mathbf{t}^T K^{-1} \mathbf{t}. \quad (6)$$

The above state is a generalization of the one-component Laughlin state

$$\Psi_m(z) = \prod_{a < b} (z_a - z_b)^m \exp \left(- \sum_a |z_a|^2 \right). \quad (7)$$

The $SL(N, \mathbb{Z})$ transformation (4) on A_I takes the following form on \mathbf{t} and K

$$\mathbf{t} \longrightarrow S\mathbf{t}, \quad K \longrightarrow SKS^T, \quad S \in SL(N, \mathbb{Z}). \quad (8)$$

This transformation leaves the filling fraction invariant. The state (5) is characterized by the $SL(N, \mathbb{Z})$ equivalence classes of K where the equivalence relation is

$$K \sim K', \quad \text{iff } K' = SKS^T, \quad S \in SL(N, \mathbb{Z}). \quad (9)$$

One of the quantum numbers which can be used to characterize the topological order of the state (5) is the ground state degeneracy on the torus [9, 4]

$$D = \det K. \quad (10)$$

Here we limit ourselves to the double-layer FQH states, where in this case the matrix K is a 2×2 matrix subject to the $SL(2, \mathbb{Z})$ transformation (4). The topological order (10) is an invariant of the K matrix under (4), i.e, it's the same for all members of the $SL(2, \mathbb{Z})$ equivalence class. Since $S = \pm I$ in (9) acts trivially, then only the group $PSL(2, \mathbb{Z})$ acts effectively. Now we can talk of the equivalence classes of K with $\det(K) = D$ under the action of $PSL(2, \mathbb{Z})$. We simply take all the matrices which are in the same orbit of K under $PSL(2, \mathbb{Z})$ as one equivalence class. The set of $PSL(2, \mathbb{Z})$ classes is defined as [3]

$$Cl(D) = \left\{ K = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid D = ac - b^2 > 0, a > 0 \right\} / \sim SL(2, \mathbb{Z}). \quad (11)$$

We assume that the greatest common divisor $\gcd(a, b, c) = 1$ which restricts us to primitive equivalence classes.

¹The filling fraction is the ration of the electron density to the flux density.

The set $Cl(D)$ is finite and its cardinality will be denoted by $h(D) = |Cl(D)|$. The elements in $Cl(D)$ will be written as

$$Cl(D) = \{\mathcal{C}_1, \dots, \mathcal{C}_{h(D)}\}. \quad (12)$$

We will denote the matrix K by $K(a, b, c)$ and the equivalence class of K under $PSL(2, \mathbb{Z})$ will be denoted by $\mathcal{C} = [K(a, b, c)]$.

There is a binary operation which can be defined on the set $Cl(D)$ and it turns $Cl(D)$ into a finite Abelian group. This operation is called the Gauss product (see [3]) and it takes two equivalence classes in $Cl(D)$ and produces a third equivalence class with same determinant.

Let $\mathcal{C}_1 = [K_1(a_1, b_1, c_1)]$ and $\mathcal{C}_2 = [K_2(a_2, b_2, c_2)]$ be two such equivalence classes. We say that two quadratic forms $K_1(a_1, b_1, c_1) \in \mathcal{C}_1$ and $K_2(a_2, b_2, c_2) \in \mathcal{C}_2$ are concordant if $a_1 a_2 \neq 0$, $\gcd(a_1, a_2) = 1$ and $b_1 = b_2$. Then the Gauss product of $\mathcal{C}_1 \star \mathcal{C}_2 = \mathcal{C}_3$ is defined as

$$[K_1(a_1, b, c_1)] \star [K_2(a_2, b, c_2)] = [K_3(a_3, b, c_3)], \quad (13)$$

where $a_3 = a_1 a_2$, $b_3 = b$, and $c_3 = \frac{b^2 + D}{4a_1 a_2}$. It is important to mention that any pair of quadratic forms of the same discriminant can be $SL(2, \mathbb{Z})$ -transformed to a concordant pair.

The identity element $\mathbf{1}_D$ of $Cl(D)$ with respect to the product \star is represented by

$$\mathbf{1}_D = \begin{cases} [1, 0, \frac{D}{4}]; & \text{if } D \equiv 0 \pmod{4} \\ [1, 0, \frac{1+D}{4}]; & \text{if } D \equiv 1 \pmod{4}. \end{cases} \quad (14)$$

Now that we defined the Gauss product, we can use it to show that the set of double-layer FQH states (5) is a finite Abelian group \mathcal{G} of order $h(D)$. As we explained before, the double-layer FQH states are characterized by an $SL(2, \mathbb{Z})$ equivalence class $[K]$. We define the following product on the set of double-layer FQH states:

$$\Psi_{[K_1]}(z) \odot \Psi_{[K_2]}(z) = \Psi_{[K_3]}(z), \quad (15)$$

where K_3 is given by the Gauss product of $[K_1]$ and $[K_2]$

$$[K_3] = [K_1] \star [K_2]. \quad (16)$$

The group property of \odot follows directly from the group property of \star . As far as we know, this kind of group structure on the set of FQH states didn't appear before in the study of the FQHE. The dimension of \mathcal{G} is equal to the number of equivalence classes of K with determinant $D = \det(K)$ which is nothing but the number $h(D)$. As we have seen in (10), D gives the degeneracy of the ground

state of the FQH systems which is a measure of topological order [8, 9, 11]. We propose the number $h(D)$ as another index which characterizes the topological order of the double-layer FQH states. One can use the group property of \mathcal{G} to generate new FQH states (with the same topological order and different filling fractions) from older ones. For example, a new state $\Psi_{[K_3]}(z)$ can be constructed out of two known states $\Psi_{[K_1]}(z)$ and $\Psi_{[K_2]}(z)$ by simply multiplying them using \odot . Whether these new states have been constructed before or realized experimentally is unknown to the author.

We recall the *fundamental theorem of finite Abelian groups* which states that every finite Abelian group is isomorphic to a direct product of cyclic groups of prime-power order, where the decomposition is unique up to the order in which the factors are written. Using this fact, we can write the order- $h(D)$ Abelian group \mathcal{G} as

$$\mathcal{G} = \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_n^{n_n}}, \quad (17)$$

where

$$h(D) = p_1^{n_1} \times \cdots \times p_n^{n_n}. \quad (18)$$

This implies that the set of FQH state enjoys a set of discrete symmetries which emerges entirely from the dynamics. This is not a symmetry of the underlying Hamiltonian but a property of the K -matrix characterization of the FQH states. Since K is related to the dynamical motion of the electron, the symmetry \mathcal{G} is dynamical.

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